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73. On River Bed Variations and Stable Channels in Alluvial Streams

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Abstract

As a first step of investigating river bed variations and stable channels in alluvial streams, the present paper deals with a theoretical consideration on the river-bed variation and on the hydraulic characteristics in a stable channel with uniform grain-size under constant discharge. On the other hand, experiments for stable channels through constrictions are carried out. Hydraulic characteristics in a abrupt expansion with a movable bed are made clear, and longitudinal profiles and cross-sectional forms in the stable state are obtained. The longitudinal profiles show good agreements with the theoretical ones. The cross-sectional forms are discussed with respect to the secondary flow.

1. Introduction

Stream bed variation will occur when river improvement works such as widening the river width, constructing cut-offs, and constructing dams, are designed. In some cases the variations will exercise a large influence on flood-control and water-utilization. Therefore, the prediction of the stream bed variation is now considered as one of the most important problems of river engineering.

Stream bed variations can be classified as follow: (a) The variation to which one-dimensional analysis can be applied: one-dimensional variation (b) The variation to which two-dimensional analysis must be applied: two-dimensional variation. Longitudinal variation of mean bed belongs to the former type, and the variation in a cross-section or local scour around a hydraulic structure belongs to the latter type. Both types have very different characteristics. Even in the case of (a), there are various kinds of stream-bed variations from the small sand ripple to the sediment deposition in a great reservoir.

In the case of variations of large scale compared with the water depth, the variation will occur gradually and the acceleration term with respect of t in the equation of motion for flow may be neglected. On the other hand, for the variations of small scale such as the sand ripples the acceleration term can not be neglected.

Dr. Iwagaki[1,2] proposed the method of characteristics for analysing the mechanism of stream bed variations of large scale, and made clear their mechanism. Moreover, he treated the stability of stream bed variations and pointed out that river-bed is generally stable for the small variations.

Dr. Matsunashi[3] pointed out by the mathematical treatment of the equation including acceleration term that the small variation could not be stable for some flow and sediment conditions ; this fact may correspond to the occurence of sand ripple. On the other hand, two-dimensional variation has hardly been treated up to the present day.

In this paper, the river-bed variations and stable channels in alluvial streams were treated theoretically and experimentally.

In chapter 2, one-dimensional analysis of river bed variation, the author discussed some problems in the treatment of the characteristics equation for the one-dimensional variation of large scale obtained by Dr. Iwagaki, and moreover the author obtained an analytical solution for the stream-bed variation in a uniform channel by the small amplitude theory.

In chapter 3, the characteristics of stable channel for uniform flow were discussed.

In chapter 4, the author treated the stable channels for varied flow.

Stable profile in a channel having a gradually varied width has been studied by many researchers[4~7], and various kinds of equations expressing the equilibrium bed slope have been proposed, but they are expressed in a form which is too complicated to apply to a natural stream where various affecting factors may exist. Therefore, it will be necessary to make clear the characteristics in the equilibrium state and to apply them in obtaining the stable profile.

The author discussed these problems and obtained a new method to express the stable profile.

In an abrupt expansion, separation of flow may occur and the eddies or rolles may be generated, which may have an important effect on the stream-bed variation. Gross-sectional forms in an alluvial stream for varied flow may have two-dimensional characteristics.

Those problems require two-dimensional treatment which is considered a very difficult problem. Therefore, the author treated them experimentally and made clear some characteristics.

2. One-dimensional analysis of river bed variation

(1) Fundamental equation and its analysis.

Fundamental equation for the analysis of river bed variation must be derived from the equations of motion and continuity for the water and for the sediment. The most important problem in the analysis may be to establish the resistance law in an alluvial channel and the rate of sediment transportation, which have been studied by many researchers. A number of experimental and theoretical formulas have been proposed up to the present but few of them are applicable to natural steams.

There are two kinds of methods to solve the fundamental equations, that is, by characteristics and by finite intervals. The former is more convenient than the latter to examine the characteristics of river bed variation. Let me demonstrate the method by characteristics with some discussions.

In general, the flow in natural stream can be approximately assumed to be a steady flow for the analysis of the river bed variation because the variation will occur gradually.

In the above case, let x be the abscissa axis along the bed surface towards the downstream, and h the water depth vertical to x axis, and assuming that the coefficient for velocity distribution α is equal to 1.0, the equations of motion and continuity for flow are expressed as follows respectively,

$$\frac{dh}{dx}=i-\frac{d}{dx}\left(\frac{v^2}{2g}\right)-\frac{u_*^2}{gR} \tag{1}$$

$$A\cdot v=Q \tag{2}$$

in which i is the bed slope, and R the hydraulic mean depth, and the u_* shear velocity.

Although many formulas have been proposed for the rate of bed load transportation, the following formula of Kalinske-Brown type is now adopted:

$$q_B=\alpha' u_*(u_*^2-u_{*c})^m \tag{3}$$

$$\alpha'=Kd/[\{(\sigma/\rho)-1\}gd]^m \tag{4}$$

where q_B is the rate of transport in volume of material per unit time per unit width of section, and K and m are constants.

When u_* is much larger than u_{*c} and $K=10$, $m=2$, Eq. (3) becomes to the formula proposed by Brown. Denoting the elevation of river bed upwards from a datum surface Z, the equation of continuity in respect to the sediment transport in a rectangular channel with a uniform width B is given by

$$\frac{\partial Z}{\partial t}+\frac{1}{B(1-\lambda)}\frac{\partial(q_B\cdot B)}{\partial x}=0 \tag{5}$$

where λ is the porosity of the sediment.

The slope of river bed i is expressed by

$$i=i_0-\frac{\partial Z}{\partial x} \tag{6}$$

where i is the slope of the datum surface.

By the use of Manning formula, $u_*^2/gR=n^2v^2/R^{4/3}$, u_* may be expressed for a wide rectangular channel by the following equation

$$u_*=g^{1/2}\cdot n\cdot Q/h^{7/6}\cdot B \tag{7}$$

If the datum surface is replaced with the river bed surface, treatment of the above equations will yield

$$\frac{dx}{dt}=A'B' \tag{8}$$

$$\frac{dZ}{dt}=A'\left(B'\frac{dh}{dx}+C'\frac{dB}{dx}\right) \tag{9}$$

It shows that Eq. (9) holds on the characteristic curves expressed by Eq. (8). In Eq. (9),

$$\left.\begin{aligned} A'&=\frac{\alpha' n^{2m+1}g^{m+(1/2)}Q^{2m+1}}{(1-\lambda)B^{2m+1}h^{13/6}}\left(\frac{1}{h^{7/3}}-\frac{1}{h_K^{7/3}}\right)^{m-1} \\ B'&=\frac{7}{6}\left\{\left(\frac{1}{h^{7/3}}-\frac{1}{h_K^{7/3}}\right)+\frac{2m}{h^{7/3}}\right\}, \quad C'=\frac{2m}{Bh^{4/3}} \end{aligned}\right\} \tag{10}$$

and, h_K is the depth for critical tractive force and may be shown by the critical shear velocity u_{*c} as,

$$h_K = (n \cdot g^{1/2} \cdot Q/Bu_{*c})^{6/7} \tag{11}$$

Eq. (9) obtained by Dr. Iwagaki may be used conveniently for the analysis of river bed variation.

But the numerical calculation of it requires much labors, so the use of an electoronic computer is desired in practics.

Although the datum surface is always replaced with the river bed surface in the above treatment, it may be considered to be convenient for the calculation by an electronic computer that the datum surface is not changed with respect to time. With these points in mind, the author will reduce a fundamental equation for river bed variation convenient for the calculation by an electronic computer.

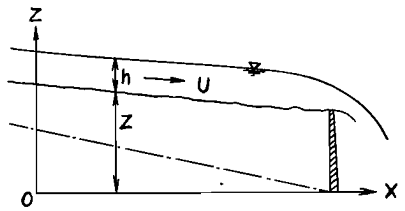

Fig. 1. Definition sketck.

With reference to Fig. 1, equation of motion for non-uniform flow in a wide rectangular channel is

$$\frac{dh}{dx} = -\frac{\partial Z}{\partial x} - \frac{d}{dx}\left(\frac{v^2}{2g}\right) - \frac{u_*^2}{gh} \tag{12}$$

Equation of continuity of flow is

$$B \cdot h \cdot v = Q(=\text{const.}) \tag{13}$$

As described before, a number of theoretical and experimental formulas have been proposed. In the following analysis the author will adopt a formula proposed by Dr. Sato, Dr. Kikkawa and Dr. Ashida[8] instead of Eq. (3).

The formula which was obtained by a great number of their experiments, and those of Gilbert, with theoretical considerations are expressed as follows:

$$q_B = \frac{\varphi u_*^3}{(\sigma/\rho - 1)g} F\left(\frac{u_{*c}^2}{u_*^2}\right) \tag{14}$$

in which φ is a coefficient depending on roughness coefficent and was determined as the following function of Manning's n by the experimental data

$$n \geq 0.025 \quad \varphi = 0.62$$
$$n \leq 0.025 \quad \varphi = 0.62\ (40n)^{-3.5}$$

This means that the less the roughness coefficient becomes the more the rate of bed load transportation increases.

TABLE 1.

τ_c/τ	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
F	1.0	0.997	0.978	0.927	0.835	0.697	0.527	0.355	0.212	0.111	0.048

F is a function of u_{*c}^2/u_*^2 obtained theoretically, which is shown in Table 1. Equation of continuity with respect to sediment was expressed in Eq. (5).

When the resistance law is according to Manning's equation, shear velocity and critical shear velocity are,

$$u_* = \frac{g^{1/2}\cdot n\cdot Q}{Bh^{7/6}}, \quad u_{*c} = \frac{g^{1/2}\cdot n\cdot Q}{Bh_K{}^{7/6}} \tag{15}$$

in which h_K is the depth for the critical tractive force.

Let h_0 be the depth at a reference section, i_0 the river bed slope, and q_{B0} th rate of bed load transportation per unit time per unit width.

Now let express depth, elevation of river bed, and t in the following dimensionless forms:

$$h/h_0 = \zeta \tag{16}$$

$$h/h_K = \gamma \tag{17}$$

$$Z/h_0 = \eta \tag{18}$$

$$\xi = (i_0/h_0)\,x \tag{19}$$

$$\tau = \{i_0 q_{B0}/(1-\lambda)h_0{}^2\}\cdot t \tag{20}$$

Inserting Eqs. (15), (20) into Eqs. (12) and (14), the following equations may be obtained for a uniform channels :

$$A_1\frac{\partial\zeta}{\partial\xi} - \frac{\partial\eta}{\partial\tau} = 0 \tag{21}$$

$$B_1\frac{\partial\zeta}{\partial\xi} + \frac{\partial\eta}{\partial\xi} + \frac{1}{\zeta^{10/3}} = 0 \tag{22}$$

in which,

$$\left.\begin{aligned} A_1 &= \frac{7}{\zeta^{9/2}F(\gamma^{7/3})}\left\{\frac{1}{3}\left(\frac{\gamma}{\zeta}\right)^{7/3}\cdot F'\left\{\left(\frac{\gamma}{\zeta}\right)^{7/3}\right\} + \frac{1}{2}F\left\{\left(\frac{\gamma}{\zeta}\right)^{7/3}\right\}\right\} \\ B_1 &= 1 - F_0{}^2/\zeta^3 \\ F_0{}^2 &= v_0{}^2/gh_0 \end{aligned}\right\} \tag{23}$$

Now, taking ξ' axis along the bed surface towards the upstream for practical convenience, Eqs. (21) and (22) are written as

$$A_1\frac{\partial\zeta}{\partial\xi'} + \frac{\partial\eta}{\partial\tau} = 0 \tag{21$'$}$$

$$B_1\frac{\partial\zeta}{\partial\xi'} + \frac{\partial\eta}{\partial\xi'} - \frac{1}{\zeta^{10/3}} = 0 \tag{22$'$}$$

These are represented by the characteristic equations

$$\left.\begin{aligned} &d\tau=0 \\ &B_1\frac{d\zeta}{d\xi'}+\frac{d\eta}{d\xi'}-\zeta^{-10/3}=0 \end{aligned}\right\} \tag{24}$$

$$\left.\begin{aligned} &\frac{d\tau}{d\xi'}+\frac{B_1}{A_1}=0 \\ &\frac{d\eta}{d\xi'}-\zeta^{-10/3}=0 \end{aligned}\right\} \tag{25}$$

In order to estimate the river bed variation it is necessary to solve these equations under the given initial and boundary conditions.

Initial condition may be given in the form

$$\tau=0: \quad \eta=f(\xi') \tag{26}$$

On the other hand it seems very difficult to give the boundary condition appropriately in a alluvial channel, since the variation of river bed changes the hydraulic condition.

It is considered that the relation between the energy head and the discharge at a river mouth or at a dam will be little changed by the river bed variation.

Therefore, this may be taken as a boundary condition, and is given by

$$\xi'=\xi_1': \quad \zeta+\frac{1}{\zeta^2}\frac{F_0{}^2}{2}+\eta=\text{const.} \tag{27}$$

With reference to Fig. 2, Eqs. (24) and (25) may be written in the following forms with finite increments.

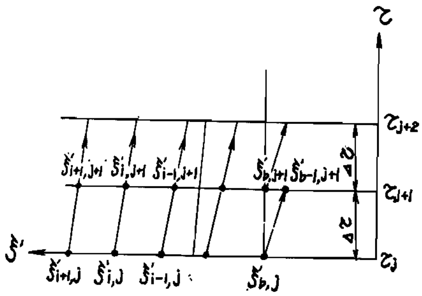

Fig. 2. Notation of step for calculation.

$$\left.\begin{aligned} &\tau_{i,j+1}-\tau_{i,j}=\Delta\tau \text{ (const.)} \\ &\zeta_{i+1,j}=\zeta_{ij}+\frac{1}{B_{i,j}}\{\zeta_{i,j}{}^{-10/3}(\xi'_{i+1,j}-\xi'_{i,j})-(\eta_{i+1,j}-\eta_{i,j})\} \end{aligned}\right\} \tag{28}$$

$$\left.\begin{aligned} &\xi'_{i,j+1}=\xi'_{i,j}-\left(\frac{A_1}{B_1}\right)_{i,j}\cdot\Delta\tau \\ &\eta_{i,j+1}=\eta_{i,j}-\zeta_{i,j}{}^{-10/3}\left(\frac{A_1}{B_1}\right)_{i,j}\cdot\Delta\tau \end{aligned}\right\} \tag{29}$$

The author has applied the above method to the calculation of the river bed variation upstream of a dam, which was already filled up with the deposited sediment, by the use of an electroric computer.

By the results of the calculation[19] it was made clear that when the values of $\Delta\xi'$ and $\Delta\tau$ were taken too large the calculation-unstability occured and the solutions became oscillating.

After some trials and considerations, it was concluded that the best way to prevent the calculation-unstability, which would be considered one of the most important problems in the calculation, was to use the mean values at the beginning and at the end of $\Delta\xi'$ and $\Delta\tau$.

For example, $(A_1/B_1)_{i,j}$ in Eq. (29) must be replaced by $\{(A_1/B_1)_{i,j}+(A_1/B_1)_{i,j}+1\}/2$ which requires a few trial calculations.

(2) *Small amplitude theory for river bed variation.*

A small deviation from the stable state is considered as a special case of river bed variation, which can be treated by the small amplitude theory.

Now, putting

$$\left.\begin{aligned} h&=h_0+h' \\ v&=v_0+v' \\ Z&=Z_0+Z' \end{aligned}\right\} \tag{30}$$

in which subscripts 0 and $'$ will indicate the values in equilibrium state and the small deviations from the equilibrium state respetively.

Inserting Eq. (30) into Eqs. (1)~(7) and neglecting the small terms, the following equation with respect to Z' may be derived for a wide rectangular channel:

$$a\frac{\partial^2 Z'}{\partial x^2}+b\frac{\partial^2 Z'}{\partial x\partial t}-c\frac{\partial Z'}{\partial x}=0 \tag{31}$$

where,

$$\left.\begin{aligned} &a=g, \quad b=6(gh_0-v_0^2)/7K'v_0h_0 \\ &c=20n^2\cdot g\cdot v_0/7K'h_0^{7/3} \\ &K'=\{\alpha'\cdot u_*/(1-\lambda)v_0h_0\}\{(u_*^2-u_{*c}^2)^m+2mu_*^2(u_*^2-u_{*c}^2)^{m-1}\} \end{aligned}\right\} \tag{32}$$

Now, putting,

$$Z'=Z\cdot e^{\delta x-\gamma t} \tag{33}$$

following equation may be derived from Eq. (31):

$$a\frac{\partial^2 Z}{\partial x^2}+b\frac{\partial^2 Z}{\partial x\partial t}+\frac{\partial Z}{\partial x}(2a\delta-b\gamma)+\frac{\partial Z}{\partial t}(b\delta-c)+Z(c\gamma-b\gamma\delta+a\delta^2) \tag{34}$$

in which γ and δ can be determined arbitrarily.

In order to eliminate the terms of $\partial Z/\partial x$ and $\partial Z/\partial t$ in Eq. (34), γ and δ are choosen as follows,

$$\delta=\frac{c}{b} \tag{35}$$

$$\gamma = \frac{2ac}{b^2} \tag{36}$$

Introduction of Eqs. (35) and (36) into Eq. (34) will be found to yield

$$a\frac{\partial^2 Z}{\partial x^2} + b\frac{\partial^2 Z}{\partial x \partial t} + \frac{ac^2}{b^2} Z = 0 \tag{37}$$

And, putting

$$\xi = t - \frac{b}{a} x \tag{38}$$

$$\eta = t \tag{39}$$

Eq. (37) becomes

$$\frac{\partial^2 Z}{\partial \xi \partial \eta} - \frac{a^2 c^2}{b^4} Z = 0 \tag{40}$$

This is an equation of hyperbolic type and may be integrated by Rieman's method.

The author now integrates Eq. (40) under the following initial and boundary conditions:

$$t = 0: \quad Z' = f(x) \tag{41}$$

$$x = 0: \quad Z' = 0, \quad \partial Z'/\partial x = 0 \tag{42}$$

which are rewritten by ξ, η and Z as follows,

$$\eta = 0, \quad \xi = -\frac{b}{a} x: \quad Z = f\left(-\frac{a}{b}\xi\right) \cdot e^{\frac{a\delta}{b}\xi} \tag{43}$$

$$\xi = \eta: \quad Z = 0 \quad \frac{\partial Z}{\partial \eta} - \frac{\partial Z}{\partial \xi} = 0 \tag{44}$$

Rieman's function of Eq. (40) will be given by

$$\frac{\partial^2 v}{\partial \xi \partial \eta} - \frac{a^2 c^2}{b^4} v = 0 \tag{45}$$

Now, putting

$$w = (\xi - \xi_0)(\eta - \eta_0) \tag{46}$$

$$2\sqrt{-\frac{a^2 c^2}{b^4}} \cdot w = S \tag{47}$$

and, considering v as a function of w alone, Eq. (45) may be written as

$$\frac{d^2 v}{dS^2} + \frac{1}{S}\frac{dv}{dS} + v = 0 \tag{48}$$

which is the equation of Bessel and will be integrated as

$$v = AJ_0(S)$$

Taking A as v will be unity as $s = 0$, the above Eq. will be reduced as

$$v = J_0(S) \tag{49}$$

Following integral along the closed integral-pass in $\xi-\eta$ plane will be

$$\phi_c(P d\eta - Q d\xi) = 0 \tag{50}$$

in which

$$P = \frac{1}{2}\left(v\frac{\partial Z}{\partial \eta} - Z\frac{\partial v}{\partial \eta}\right) \tag{51}$$

$$Q = \frac{1}{2}\left(v\frac{\partial Z}{\partial \xi} - Z\frac{\partial v}{\partial \xi}\right) \tag{52}$$

Integrating along the integral-pass as shown in Fig. 3 and converting Z, ξ and η into Z', x and t, the following equation will be obtained:

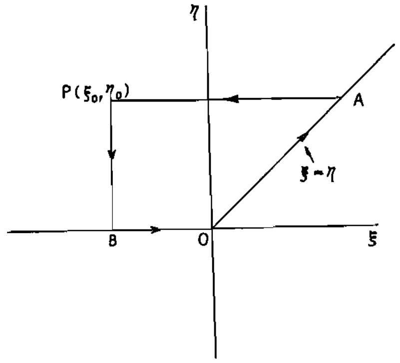

Fig. 3.

$$Z'(x_0, t_0) = f(x_0)e^{-\gamma t_0} - e^{\delta x_0 - \gamma t_0}\int_0^{x_0} f(x)e^{-\delta x}\frac{\sqrt{\frac{ac^2}{b^3}t_0}}{\sqrt{x_0 - x}} J_1\left\{2\sqrt{t_0\frac{ac^2}{b^3}(x_0 - x)}\right\}dx \tag{53}$$

Since the river-bed variation occurs in a limited reach which depends on the initial condition, x_0 in Eq. (53) can not be taken arbitrarily.

Suppose the river bed variation at $t=0$ occurs in $x \leq x_a$ since the celerity of the variation is (a/b) in Eq. (31), the river bed variation at $t=t_0$ may occur in the following extent of x_0:

$$x_0 \leq x_a + (a/b)t_0$$

The second term in Eq. (53) becomes the largest at $x_0 = x_a + (a/b)t_0$, when the term varies with respect to t_0 as $e^{-(ac/b^2)t_0}\sqrt{t_0}$. Since a and c are always positive, $e^{-(ac/b^2)t_0}\sqrt{t_0}$ becomes to 0 with times t_0.

Since γ is always positive, the first term in Eq. (53) becomes to 0 for large values of t_0.

Therefore, it may be concluded that a small deviation from the stable state always decreases, which process may be expressed by Eq. (53).

For small t_0 when $2\sqrt{t_0(ac^2/b^3)(x_0-x)}$ is about $0\sim0.15$, the following approximation will be expressed with sufficient accuracy:

$$J_0[2\sqrt{t_0(ac^2/b^3)(x_0-x)}] \doteqdot \sqrt{t_0(ac^2/b^3)(x_0-x)}$$

and, Eq. (53) can be written in the following simple form:

$$Z'(x_0, t_0) = e^{-rt_0}\{f(x_0) - (ac^2/b^3)t_0 e^{\delta x_0}\int_0^{x_0} f(x)e^{-\delta x}dx\}$$

With knowledge concerning the stable state, Eq. (53) or (54) may by applicable to discuss the effects of dredging and artificial works on river bed variation.

3. Characteristics of stable channels for uniform flow.

It is very important to make clear the characteristics of stable channels for uniform flow and grain size distribution, because they are considered to give a basis of stable channels in general cases.

In the case of a wide rectangular channel with constant width, the following expressions will be obtained from Eq. (7),

$$h = (n\sqrt{g}Q/u_*B)^{6/7} \tag{55}$$

$$i = (n^2/h^{10/3})(Q/B)^2 \tag{56}$$

in which u_* will be decided depending on the given rate of sediment transportation.

According to the formula for bed load transportation proposed by Dr. Sato, Dr. Kikkawa and Dr. Ashida which was expressed by Eq. (14), u_* may be written as

$$u_* = \{[(\sigma-\rho)g/\rho\cdot\varphi F(\tau_c/\tau)](Q_B/B)\}^{1/3} \tag{57}$$

in which Q_B is the rate of bed load transportation per unit time across a section.

Since the depth h and slope i given by Eqs. (55) and (56) are the values in an equilibrium state for uniform flow, they may be called equilibrium uniform depth and slope respectively. The sole value of equilibrium uniform depth and slope are determined by Q, Q_B, B and d (grain-size diameter). Although Manning roughness coefficient n also relates to h and i, it is considered in an alluvial stream to be determined by Q, Q_B, B and d, and therefore it may be included in them.

It will be found later that an equilibrium bed configuration in a channel with a varied width can be easily obtained by use of the equilibrium uniform depth and slope. Therefore it is very important in those problems to make clear the characteristics of equilibrium uniform depth and slope, which will be shown as follows.

The relation between specific energy with reference to the river bed H and depth h for constant discharge per unit width is expressed by

$$H = h + (q^2/2g)(1/h^2) \tag{58}$$

On the other hand, the relation between H and h for constant u_* and n is expressed by

$$H = h + (u_*^2/2g^2n^2)h^{1/3} \tag{59}$$

It is seen from Eq. (57) that u_* being constant may be equivalent to constant sediment discharge per unit width q_B and constant grain-side diameter d.

Eq. (58) and (59) are represented by two kinds of curves as shown in Fig. 4 (a) and (b) which have quite different characteristics.

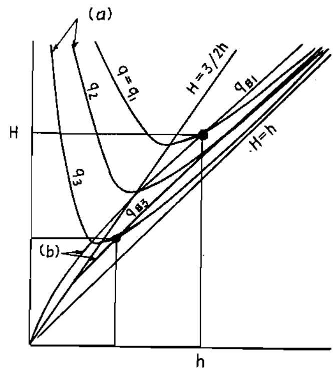

Fig. 4. Relation between H. (Specific Energy) and h (depth) in the case of q=const. and q_B=const.

There exists a max. discharge possible for a given specific energy H. On the other hand sediment discharge may possibly take any amount for a given value of H, as the relation between H and h for constant value of q_B is represented by a group of curves passing through origin.

Equilibrium uniform depth and specific energy for given values of q and q_B are obtained by the abscissa and the ordinate of an intersecting point of two curves for the values of q and q_B. It is seen from Fig. 4 that only one such an intersecting point ever exists. On the other hand there exist two kinds of depth, supercritical and subcritical, for constant q and H in the case of a rigid bed without sediment transportation. From this fact the phenomena in a channel with a movable bed seem to be quite different from those in a channel without sediment transportation.

Now, the author will exame how the equilibrium uniform depth and slope vary with channel width B in the case of constant Q, Q_B and d.

Representing the quantities at a reference section with suffix 0, and putting $u_{*c}/u_{*0} = x$, $u_*/u_{*0} = y$, Eq. (57) will yield

$$\frac{B}{B_0} = \frac{F(x^2)}{F(x^2/y^2)} \frac{1}{y^3} \tag{60}$$

which is plotted with a parameter of x as shown in Fig. 5.

When shear-velocity u_* is much larger than critical shear-velocity, which case may be found to be common in natural streams, Eq. (6) becomes

$$y = (B/B_0)^{-1/3} \tag{61}$$

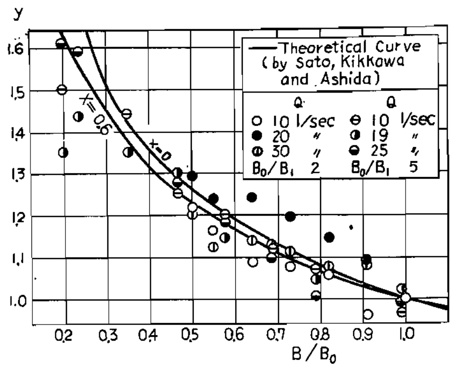

Fig. 5. Relation between $y(=u_*/u_{*0})$ and B/B_0 in the case of x=const.

From Eq. (55), equilibrium uniform depth may be written in the form

$$h/h_0=\{1/(B/B_0)\cdot y\}^{6/7} \tag{62}$$

which is plotted in Fig. 6.

Experimental values plotted in Fig. 5 and Fig. 6 were obtained by the author's experiments which will be explained in the next chapter.

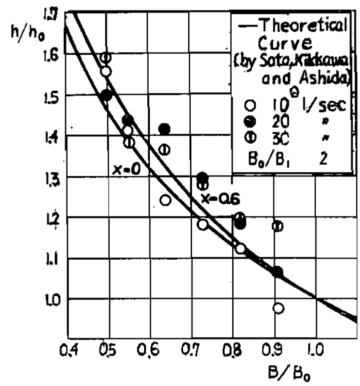

Fig. 6. Relation between h/h_0 and B/B_0 in the case of x=const.

For $u_{*c}\ll u_*$, Eq. (62) becomes

$$(h/h_0)=(B/B_0)^{-4/7} \tag{63}$$

In the same way, the following equations are obtained:

$$(A/A_0)=(B/B_0)^{3/7} \tag{64}$$

$$(F_r/F_{r0})=(B/B_0)^{-1/7} \tag{65}$$

$$(i/i_0)=(B/B_0)^{-2/21} \tag{66}$$

Where A is cross-sectional area and F_r Frounde number. It is specially noticed that the effect of B on F_r or i is very small and these values remain constant, while Q, Q_B and d are constant even though width B is changed.

Straub[10] proposed an equation of the same type as Eq. (63) for equilibrium uniform depth:

$$(h/h_0)=(B/B_0)^{-0.643} \tag{67}$$

Griffith[10] also proposed in the form

$$(h/h_0)=(B/B_0)^{-0.637} \tag{68}$$

4. Stable channels for varied flow

Two aspects which are a longitudinal profile and a cross-sectional form are considered in a stable channel for varied flow.

Equilibrium bed profile for gradually varied flow has been studied by many researchers, but little is known about the characteristics.

The author has discussed the characteristics and has proposed a new method by which the longitudinal profile of mean bed might be easily obtained from equilibrium uniform depth and slope.

Concerning equilibrium bed profiles and flow characteristics for abrupt expansion in which separation will occur, little has been studied up to the present date. The same may be said for cross-sectional forms through constrictions and expansions. In this paper, the author pointed out some interesting phenomena in these problems by his experiments.

(*1*) *Theory for equilibrium bed profiles in a gradually varied channel.*

Equation of motion for non-uniform flow in a wide rectangular channel is

$$\frac{dz}{dx}+\frac{dh}{dx}+\frac{d}{dx}\left(\frac{v^2}{2g}\right)+\frac{u_*^2}{gh}=0 \tag{1}$$

Manning formula for resistance law is

$$\frac{u_*^2}{gh}=\frac{n^2\cdot v^2}{h^{4/3}}=1 \tag{69}$$

Integration of Eq. (1) yields

$$Z_x+H_x=H_{e0}-\int_0^x I dx \tag{70}$$

in which Z_x is the elevation of the river bed above a datum plane at any section, H_x the specific energy refering to the river bed at the section and H_{e0} the energy head above the datum plane at the place where the boundary condition is given. As far as the law of resistance and of sediment transportation in a gradually varied channel are assumed to be equal to those in a uniform channel, the depth and the energy gradient will be equal to the equilibrium uniform depth and slope in the uniform channel having the same width as that of the gradually varied channel at each section respectively.

Therefore, the specific energy H and energy gradient I in Eq. (70) may be easily obtained by the values of Q, Q_B and the width at section x.

As described before, the equilibrium uniform slope, and therefore the energy gradient I, remain almost constant while the Q and Q_B are constant even though the width varies. This means that the right side in Eq. (70)

is almost constant no matter how the width varies.

From this fact it may be found that the energy equation given by Eq. (70) is very convenient to estimate the stable profile in a gradually varied channel and the energy head H_{e0} at the boundary is a very important factor in obtainig a stable profile.

(2) *Experiments for stable channels.*

For the purpose of verifying the above theoretical treatments, and to make clear the two dimensional characteristics in a stable channel, the experiments were conducted in a concrete channle of 1 m in width and 14 m in length in which two kinds of constrictions were set up and the experimental sand of 30 cm in thickness was laid as shown in Fig. 7.

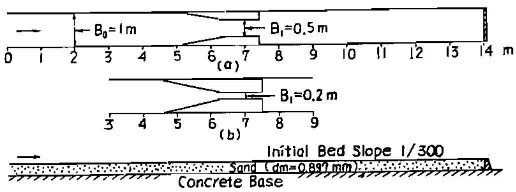

Fig. 7. Experimental flume.

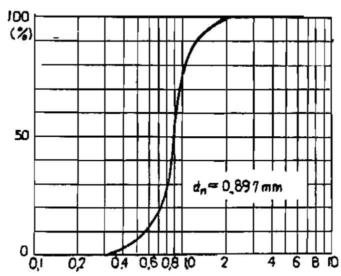

Fig. 8. Grain-size accumulation curve of sand used.

The grain-size of the sand used is comparatively uniform and the mean diameter is 0.897 mm as shown in Fig. 8.

The experiments were carried out in the extent shown in Table 2.

TABLE 2.

No. of experiments	1	2	3	4	5	6	7	8	9	10
Expanding ratio B_0/B_1	2	2	2	5	5	5	5	5	5	5
Discharge (l/s)	10	20	30	10	10	10	10	10	19	25
Initial bed slope	—	—		1/300		—	—	—	—	—

In the exqeriment with a movable bed it is necessary to use a sand-feeder with sufficient accuracy, but in this experiment such a sand-feeder was not used ; therefore the author fed the sand in such a way so as not to change the water level at the upper end which would be attained soon after the beginning of flow.

Water level and bed configuration such as longitudinal profile and cross-sectional form for the equilibrium state were measured. Moreover the hydraulic characteristics in an abrupt expansion were examined.

(**b**) variation of water level.

The profile of water level through a constriction initially takes a form, being dammed up and nearly horizontal upstream of the constriction, as shown in Fig. 9.

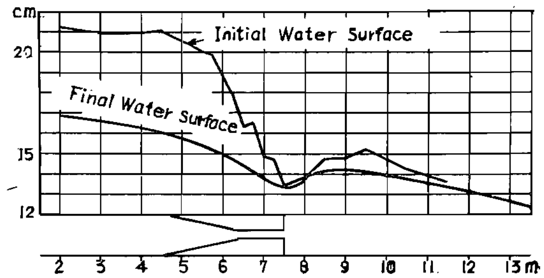

Fig. 9. Initial and final water surface profile through constriction.

But as the bed at the constriction will be scoured soon after the beginning of flow, the water surface becomes to be smoothed. This variation takes place rapidly at the first step and after that gradually occurs as shown in Fig. 10 which shows the change of water level with time in the case of x = const.

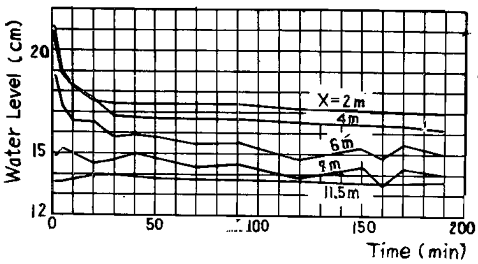

Fig. 10. Change of water level with time in the case of x=const.

(**c**) Hydraulic characteristics in an abrupt expansion with movable bed.

It is one of the most important characteristics in an abrupt expansion that flow separate from the wall and rollers or wakes are generated. It may be also one of the most important characteristics in an abrupt expansion that the scoured sediment along the narrow reach deposits at the successive expansion and forms sand ridges as shown in Fig. 11. Both the flow directions at the water surface and near the bed, which were obtained after the sand ridge was formed, are almost equal as shown in Fig. 12. On the other hand sand movement was confined to the middle part of the channel.

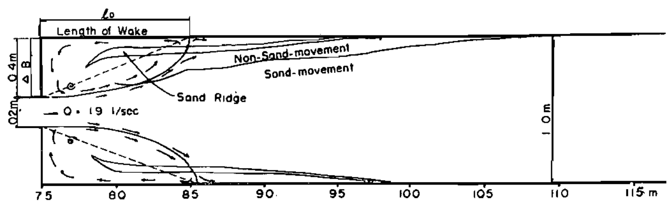

Fig. 11. Wake and sand ridge in an abrupt expansion with movable bed.

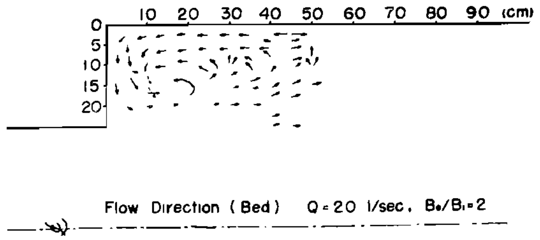

Fig. 12(a). Flow direction in an abrupt expansion (water surface).

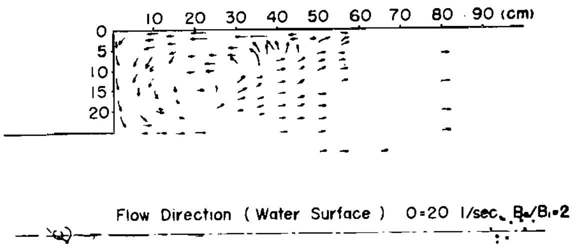

Fig. 12(b). Flow direction in an abrupt expansion (Near the bed).

In order to investigate the effects of sand ridges on the wake the author also conducted the experiments in which the sand ridges were removed. In that case the sand ridges were not formed again, which would imply that the sand ridges were formed by the scouring sediment at the narrow reach. It may be found from Fig. 13(a) that the sand ridge has no influence on the form of the boundary between the wake and the main flow. The boundary also does not vary with discharge as shown in Fig. 13(b). But the downstream end of the boundary is not stable and fluctuated case by case.

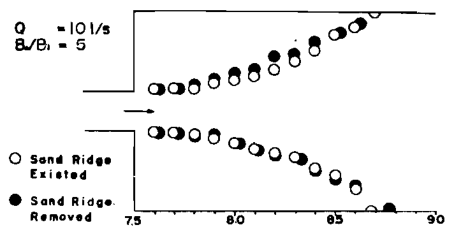

Fig. 13(a). Influence of the sand ridge on the boundary between wake and main flow.

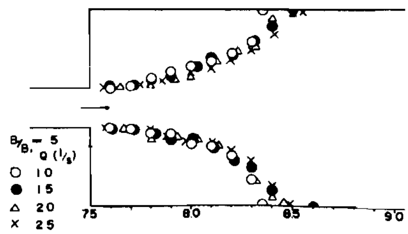

Fig. 13(b). Boundaries between wake and main flow for various discharge.

One of the characteristics in an abrupt expansion may be an asymmetrical flow which has been observed in the experiment with fixed bed. But in the case of movable bed, the flow was symmetric and the wakes in both sides were almost equal, which meant the flow in a channel with movable bed might be more stable than that with fixed bed.

Denoting the length of wake l_D and the values of $\Delta B/l_D$ obtained in the experiment were from 0.40 to 0.47 as shown in Table 3, in which h is depth, Fr Froude number, and subscripts 1 and 2 indicate the values at the reference sections upstream and downstream of the wake respectively. On the other hand the values of 0.16~0.18 were obtained for an abrupt expansion with fixed bed by the author's experiment[11].

TABLE 3.
Length of wake.

No.	Q(l/s)	B_0/B_1	h_1(cm)	h_2(cm)	Fr_1	Fr_2	$\Delta B/l_D$	
1	10	5	8.55	2.64	0.647	0.745	Right L ift	0.47 0.47
2	15	〃	10.80	3.61	0.680	0.700	R. L.	0.37 0.40
3	20	〃	15.15	4.77	0.547	0.613	R. L.	0.42 0.41
4	25	〃	20.15	6.35	0.448	0.505	R. L.	0.41 0.38
5	10	2	4.36	2.29	0.701	0.922	R. L.	0.46 0.49
6	15	〃	6.09	3.69	0.640	0.675	R. L.	0.36 0.46
7	20	〃	6.81	4.42	0.719	0.689	R. L.	0.46 0.38
8	25	〃	9.03	5.58	0.589	0.605	R. L.	0.51 0.43

This means the length of wake in an expansion with movable bed is much shorter than that with fixed bed. It is considered that the length of wake depends on various factors in which the slope of the water surface at an expansion may have the greatest effect.

According to the author's experiment, the relation between the expanding angle of flow, which relates to the wake length, and the slope of the water surface was as follows. In the case of the horizontal water surface, the expanding angle of flow was nearly equal to that obtained in two dimensional jet by Tollmien. In the case of the water surface with an adversed slope, which is due to conversion of velocity head to potential head and usually occurs in the case of tranquil flow through an abrupt expansion with fixed bed, expanding angle became much smaller than that for two dimensional jet. On the other hand, in the case of the water surface with positive slope, the fact was the reverse. In both cases, the greater the slope of the water surface was the greater the deviation of the expanding angle from that in two-dimensional jet.

Since the cross-section in the movable bed changes and approaches uniformity by the action of scouring and deposition, the adversed slope of the water surface which would be obtained on the fixed bed, will come to be decreased or to be slightly positive on the movable bed. Therefore the expanding angle of flow on a movable bed will be greater than that on a fixed bed. This means that the length of wake on a movable bed will be shorter than that on a fixed bed.

Cross-sectional forms.

Cross-sectional forms in an abrupt expansion vary due to the expansion of flow towards the downstream as shown in Fig. 14.

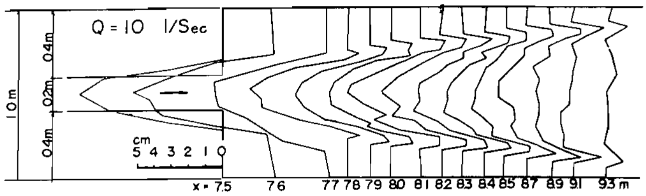

Fig. 14.

The cross-section at the beginning of the expansion has the steepest side slope equal to the angle of repose of the sand. The side slope decreases downstream as the breadth of flow increases.

(3) *Comparison between the experiments and the theory for the longitudinal profiles.*

In an abrupt expansion in which a separation of flow occurs the width of flow is not equal to that of the channel. In such a case, the width of the flow must be decided at first in order to discuss the equilibrium profile of the mean bed.

Since the velocity in the main flow rapidly increases towards the inside from the boundary between the wake and the main flow and the water in-

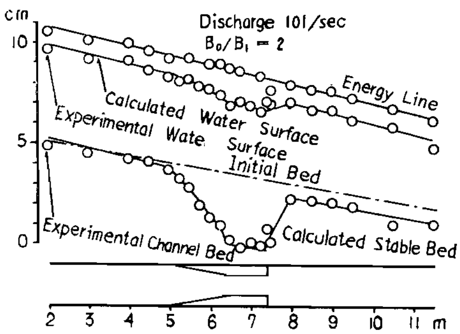

Fig. 15(a).

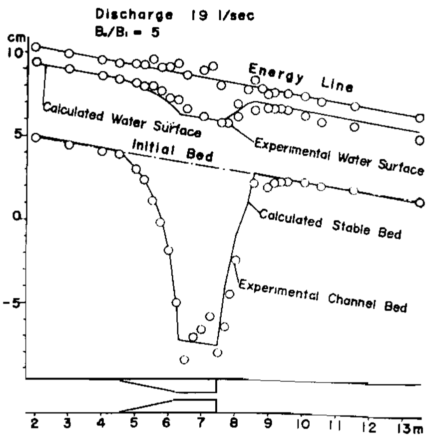

Fig. 15(b).

terchanged through the boundary is little, the width between the boundaries of both sides may be taken as the effective width. Fig. 15 represents the comparison between the theory and the experiments for the equilibrium profiles of energy line, water surface and mean bed which is calculated within the effective width.

Circular marks in Fig. 15 indicate the experimental values, and full-lines represent the values by the theory shown in 4. (1), which show a good agreement with the experiments.

It may be specially noted that the slope of energy line is almost constant in spite of the variation in width. This fact may verify the theoretical results that in the case of $u_{*c}<u_*$, the equilibrium uniform slope which is considered to be the slope of energy line in a nonuniform channel with the same width, does not vary with the variation in width while the rate of flow and sediment discharge remain constant.

The equilibrium uniform depth and the specific energy at each section can be easily obtained by Eqs. (55) and (58) using the width at the section and the given hydraulic quantities such as the rate of flow, sediment discharge and Manning's roughness coefficient. In this experiment, Manning's roughness coefficient obtained in the uniform reach was used.

Deduction of the specific energy from the energy head will yield the elevation of the sediment bed and the elevation of the water surface may be calculated by the elevation of the sediment bed plus the equilibrium uniform depth at each section.

It may be concluded that the energy equation and the equilibrium uniform depth and slope are conveniently used for obtaining the elevation of the water surface and the sediment bed.

(4) *Consideration of the cross-sectional forms.*

The stable cross-section in a non-uniform channel with movable bed takes a very complicated form depending on the variation in the channel width as shown in Fig. 16 which may be divided into five reaches.

For the first, which is the upper uniform reach, the bed form depends on how the sand will be fed. Of course the bed will be uniform for sediment

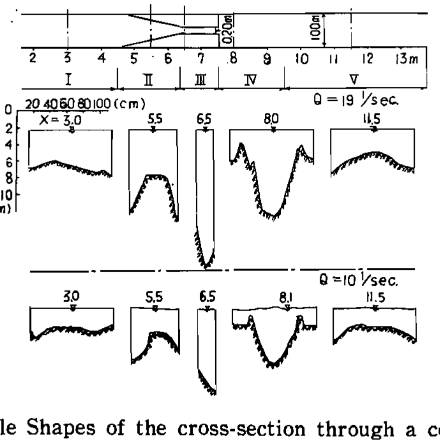

Fig. 16. Stable Shapes of the cross-section through a constriction.

uniformly fed.

For the second and the fifth, which are the reach of converging flow, the bed along the wall is much scoured and becomes much deeper than that in the middle part of the section.

Considering by the experimental fact that the water level along the wall is slightly higher than that in the middle, the secondary currents seem to exist from the wall towards the middle along the bed. Such a cross-sectional form may be due to these secondary currents.

For the third and fourth, which are the reach of diverging flow, the bed in the middle in a section is much deeper than that along the side. This may be the reverse of the converging flow.

The ratios of the depth along the left, the center, and the right side of the channel to the mean depth in a section are shown in Fig. 17 by taking a sample.

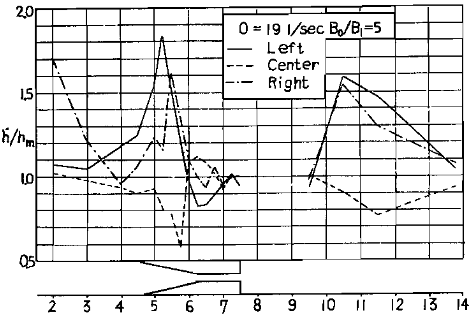

Fig. 17. Variation of h/hm through a constriction.

In a movable bed a slight converging may yield a large scouring along the wall due to the secondary effect. Therefore, it may be very important to predict the scouring depth in making a plan for revetments in such a case.

Comparison of the experimented max. depth along the side wall and the depth for critical tractive force calculated by the equation: $h_K=(gn^2Q^2/B^2u_{*c})^{3/7}$ shown in Table 4, does not always show good agreement. The dis-

TABLE 4. Comparison of max. depth along the side walls and the depth for critical tractive force.

No. of Exp.	h_{max} (cm) L.	R.	h_K (m) L.	R.
1	8.0	2.7	4.40	4.40
2	8.95	6.35	8.13	9.84
3	9.1	10.1	12.7	16.1
4	4.8	4.6	4.95	4.07
5	4.3	5.4	4.67	4.67
6	4.3	5.0	4.95	4.95
8	5.7	4.8	4.52	4.52
9	10.2	11.1	9.04	10.4

crepancy between the max. depth and the value of h_K is specially large for the left side in Run. I. This may be attributed to the fact that the flow was one-sided in that case which differed from the assumption for obtainig the value of h_K; that is, the discharge is uniformly distributed over a section. But it seems that the order of magnitude of the max. depth can be estimated by the value of h_K in general cases.

Some interesting characteristics of a cross sectional form in a non-uniform channel with movable bed have been made clear, but it should be further investigated in connection with the secondary flow.

5. Conclusion

In the introduction to this paper, some problems in the river bed variations and stable channels in an alluvial stream were briefly discussed, based on the classification of these problems. In the second chapter, one-dimensional analysis of river-bed variation were discussed and may be summarized as follows:

(1) The equations for the river-bed variation applicable to the calculation by the use of an electroric computer were derived and the problems in the calculation were discussed.

(2) Small variations which would be given in the stable state were analized by Rieman's method and the solution representing the process to the stable state was obtained. Moreover it was made clear that the small variation may be stable in general.

In the thired chapter, the characteristics of stable channels for uniform flow were discussed. In the fourth chapter, two aspects of stable channel for varied flow which are a longitudinal profile and a cross-sectional form were discussed. From the theoretical and experimental consideration, the following summary and conclusion may be offered:

(1) The sole value of equilibrium uniform depth, slope and specific energy are determined by the values of Q, Q_B, B and d.

(2) The curves for the relation between the specific energy H and depth h for constant value of q_B are represented by the different kind of curves for const value of q. Although there exists a max. discharge possible for a given specific energy H, the sediment discharge may possibly take any amount for a given value of H.

(3) Equilibrium uniform depth and specific energy are obtained by an intersecting point of two curves only one of which ever exists for given values of q and q_B in the diagram of $H \sim h$.

(4) The author examined how the depth, slope and Froude number in the equilibrium uniform state varied with the channel width B and it was made clear that the effect of B on the slope and Froude number is very small, and therefore these values remain constant even though width B varies while Q, Q_B and d are constant.

(5) Energy equation given by Eq. (70) may be conveniently applied to estimate the stable profile in a gradually varied channel and therefore the energy head H_{eo} at the boundary may be a very important factor in obtaining a stable profile.

(6) Energy gradient in a channel with movable bed is almost constant in the equilibrium state even though the width varies while the values of Q and Q_B are constant.

(7) In an abrupt expansion in which separation of flow occurs and the width of flow is not equal to that of the channel, the width between the boundaries of the wake and the main flow in both sides can be taken for the practical purpose.

(8) The wakes formed in an abrupt expansion with movable bed are more stable and shorter than that with fixed bed.

(9) Stable cross-sections in a non-uniform channel with movable bed take very complicated forms due to the secondary flow depending on the variation in the channel width. For the converging flow, the bed along the wall is much scoured and beeomes much deeper than that in the middle part of the section. It is very difficult to estimate appropriately the max. depth, but the order of the magnitude can be estimated by the value of the depth for the critical tractive force h_K. For the diverging flow, on the contrary, the bed in the middle part in a section is much deeper than that along the side.

These problems should be further investigated in connection with the secondary flow.

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